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Study of Stochastic Differential Equations by Constructive Methods. I.

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In this work we give an algorithm to express as a convergent series the stationary averages for a class of gradient perturbations of a nonsymmetric (nongradient) Ornstein–Uhlenbeck process. The method relies on a cluster expansion in time of the Girsanov–Cameron–Martin formula for the density of the perturbed measure with respect to the Ornstein–Uhlenbeck measure. In the second paper of this series, the approach is extended to more general perturbations.

KEY WORDS: Stochastic differential equations; constructive methods; stationary measure; cluster expansion; nongradient systems, Girsanov theorem.

1. INTRODUCTION

The developments of quantum field theory and statistical mechanics in the last decades have introduced in probability theory new constructive methods which are very effective not only in proving existence theorems, but also in explicit calculations of quantities of interest. The new techniques, however, have been applied only occasionally to traditional topics in probability theory. One must also add that methods developed in connection with physical problems are often heavy and not very palatable to mathematicians.

A subject in which a constructive point of view has proved to be very useful is the theory of stochastic partial differential equations arising in various areas of natural sciences. These equations are often too singular to be approached by well-established general methods and require new ideas

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which are largely introduced on the basis of physical intuition. Renormalization theory, for example, has been instrumental in proving the existence of weak solutions for a class of reaction-diffusion equations in two space dimensions perturbed by an additive white noise in time and space.[1-3]

This is the first of a series of papers in which constructive methods from field theory and statistical mechanics will be applied systematically to the study of stochastic differential equations both ordinary and partial with the aim of computing their asymptotic stationary properties.

The construction of the invariant measure or of the equilibrium correlation functions for diffusion processes described by stochastic ordinary or partial differential equations (SODE or SPDE) is in general a difficult problem. In the case of SODE the invariant measure can be obtained in principle by solving the stationary Fokker–Planck equation (Kolmogorov forward equation), but in practice this is trivial for gradient systems with constant diffusion, while no systematic method is known for more general situations. For SPDE, the Fokker–Planck equation becomes a functional equation and the situation is even worse. The stochastic quantization of gauge theories, however, gives rise just to a problem of this kind.[5]

For the existence of stationary solutions one has general criteria like the well-known Hasminski[6] nonexplosion test. This, however, has two drawbacks. On one hand it is nonconstructive, i.e., it does not give any algorithm to calculate the stationary solutions. Then, it requires that the equations under study have strong solutions and this is seldom the case for the most interesting SPDE.

One is therefore tempted to look for more direct approaches to this important problem.

To explain our point of view let us consider as an example a system in R^n of the form

$$d\mathbf{X}_t = -A\mathbf{X}_t\,dt + \mathbf{G}(\mathbf{X}_t)\,dt + d\mathbf{W}_t \tag{1}$$

where A is an $n \times n$ matrix and $\mathbf{G}$ a smooth, but in general unbounded function of $\mathbf{X}$.

Suppose we want to calculate an expectation

$$E_{\mathbf{X}_0}(F(\mathbf{X}_T)) \tag{2}$$

where $\mathbf{X}_0$ is the initial condition and $\mathbf{X}_T$ is the solution at time T of (1). We assume for the time being that (1) has strong solutions. Suppose now that the linear part of (1), i.e., the process described by the equation

$$d\mathbf{Y}_t = -A\mathbf{Y}_t + d\mathbf{W}_t \tag{3}$$

is stable and tends to a stationary solution as $t \to \infty$. This is easily established in term of the spectral properties of A. We can rewrite (2) using Girsanov theorem (see, e.g., ref. 4)

$$E_{\mathbf{X}_0}(F(\mathbf{Y}_T)\, e^{\xi_T}) \tag{4}$$

where

$$\xi_T = \int_0^T (\mathbf{G}(\mathbf{Y}_s), d\mathbf{W}_s) - \tfrac{1}{2} \int_0^T \|\mathbf{G}(\mathbf{Y}_s)\|^2 \, ds \tag{5}$$

Now, we ask the question: is it possible to study the large-time behavior of (2) using the explicit form (4) in terms of the process $\mathbf{Y}_t$?

In this formulation our question recalls the problem of taking the infinite-volume limit in constructive quantum field theory, for wich powerful tools have been developed.[7-10]

This is the first of two papers in which we will show how the idea of cluster expansion can be applied to an expression like (4). This will yield a convergent expansion, uniformly in T, for a wide class of equations.

The main difficulty in adapting the field-theoretic techniques to an expression like (4) comes from the stochastic integral in the exponent (5). For this reason we shall consider first the simpler case in which

$$\mathbf{G} = -\nabla V \tag{6}$$

so that the stochastic integral can be explicitly performed. Let us emphasize that this choice of $\mathbf{G}$ does not imply that our system is gradient: in fact we have not supposed A symmetric. In a subsequent paper we shall allow a nongradient part in $\mathbf{G}$, but we hall still require that the global stability be ensured by a potential function V as in ref. 3.

Let us examine more closely the class of equations with $\mathbf{G}$ of the form (6) that can hopefully be discussed within our approach.

The stochastic integral in (5) can be easily performed and we obtain

$$\xi_T = -(V(\mathbf{Y}_T) - V(\mathbf{Y}_0)) + \int_0^T LV(\mathbf{Y}_s)\, ds - \tfrac{1}{2} \int_0^T \|\nabla V(\mathbf{Y}_s)\|^2 \, ds \tag{7}$$

where L is the generator of the process (3), that is,

$$LV(\mathbf{Y}) = \tfrac{1}{2} \Delta_{\mathbf{Y}} V - (A\mathbf{Y}, \nabla V) \tag{8}$$

Therefore in order that e^{ξ_T} be bounded for any finite T, it is sufficient to have, $\forall \mathbf{X}$, two positive constants c_1 and c_2 such that

$$V(\mathbf{X}) > -c_1 \tag{9}$$

$$LV - \tfrac{1}{2} \|\nabla V\|^2 = \tfrac{1}{2} \Delta_{\mathbf{X}} V - (A\mathbf{X}, \nabla V) - \tfrac{1}{2} \|\nabla V\|^2 < c_2 \tag{10}$$

It is interesting to compare (10) with the condition that V be a stochastic Lyapunov function for Eq. (1). This condition can be written

$$\tfrac{1}{2}\Delta_{\mathbf{X}} V-(A\mathbf{X}, \nabla V)-\|\nabla V\|^2<c_2-f(\mathbf{X}) \tag{11}$$

with $f>0$ and $\lim_{|\mathbf{X}|\to+\infty} f(\mathbf{X})=\infty$. This coincides with (10) if we take $f(\mathbf{X})=\frac{1}{2}\|\nabla V\|^2$ when $\|\nabla V\|^2$ diverges at infinity. However, in general (11) and (10) do not have the same content.

It is interesting to remark that (9) is a condition independent of (10). In fact (10) is compatible with potentials unbounded below, as the example for $X\in R$,

$$V(X)=-\tfrac{1}{2}\log(X^2+1) \tag{12}$$

shows.

We conclude this introduction by emphasizing the nontriviality, as far as the equilibrium measure is concerned, of the class of stochastic systems considered in this paper. The invariant measure of the process (3) is given by a Gaussian density

$$\rho_0=c\exp[-\tfrac{1}{2}(\mathbf{X}, M^{-1}\mathbf{X})] \tag{13}$$

where M is a symmetric matrix satisfying

$$AM+MA^T=I \tag{14}$$

One can be tempted to look for an invariant measure for the process (1) of the form

$$\rho=\rho_0\rho_1=c'\rho_0\exp[-2V(\mathbf{X})] \tag{15}$$

Using the Fokker–Planck equation, it is easy to see, however, that (15) can be an invariant measure only if the orthogonality condition is satisfied,

$$\nabla V\cdot[(-M^{-1}+2A)\mathbf{X}]=0 \tag{16}$$

This implies that ∇V can be different from zero only on $\ker(2A-M^{-1})$. So V must be constant except on linear subspaces of lower dimensions. Since we have assumed V at least twice differentiable, it must be constant everywhere and therefore trivial.

The stationary measure corresponding to the process (3) is the Gaussian measure generated by the covariance

$$\theta(t_1-t_2)\,e^{-A(t_1-t_2)}M+\theta(t_2-t_1)\,Me^{-A^T(t_2-t_1)} \tag{17}$$

This matrix is symmetrical in the exchange of t_1 and t_2 and the transposition of indices.

The heart of the method we propose is combinatorial and does not depend on the system being gradient or not, finite dimensional or infinite dimensional. Therefore we shall illustrate it in the simplest possible situation, that is, a one-dimensional stochastic differential equation for which the answer is known in advance. At the end we shall indicate the obvious steps to adapt the method to nontrivial cases.

2. THE MODEL AND RESULTS

We will develop our approach on a very simple one-dimensional model, i.e., the random variable ψ_s is in R. We also choose $A = 1/2$ and the drift is a third-order polynomial. More explicitly, the stochastic equation is

$$\dot{\psi}_t = -\frac{1}{2}\psi_t - \frac{\lambda}{2} :\psi_t^3: + \dot{w}_t \tag{18}$$

where $:\psi_t^4: = \psi_t^4 - 6\psi_t^2 + 3$ and $:\psi_t^3: = \psi_t^3 - 3\psi_t$.

According to the Girsanov formula, one is interested in the limit when $T \to \infty$ of quantities like

$$E_{\phi_0}^T = E_{\phi_0}(F(\phi_T)\, e^{\xi_T}) \tag{19}$$

where F is typically a polynomial and

$$\xi_T = -\frac{\lambda}{2}\int_0^T :\psi_s^3:\, dw_s - \frac{\lambda^2}{8}\int_0^T (:\psi_s^3:)^2\, ds \tag{20}$$

and ϕ_t is the one-dimensional Ornstein–Uhlenbeck process, that is the solution of $\dot{\phi}_t = -\frac{1}{2}\phi_t + \dot{w}_t$ with initial condition ϕ_0:

$$\phi_t = e^{-t/2}\phi_0 + \int_0^t e^{-(t-s)/2}\, dw_s \tag{21}$$

We will prove the following theorems.

Theorem 1. Let ϕ_0 be an arbitrary real number and F a polynomial; then, for λ small enough, independently of ϕ_0 and F, the following is true:

1. $E_{\phi_0}(F(\phi_T)\, e^{\xi_T})$ can be expressed as a convergent series uniformly in T.

2. The quantity

$$\langle F(\phi)\rangle = \lim_{T\to\infty} E_{\phi_0}(F(\phi_T)\, e^{\xi_T})$$

is expressed as a convergent series.

3. $\langle F(\phi)\rangle$ does not depend on ϕ_0.

Let us consider n polynomials $F_1,..., F_n$ and n times $t_1,..., t_n$, $t_i \geqslant 0$, $i=1,..., n$ and let $t=\sup_i t_i$. Then we have the following result.

Theorem 2. For λ small enough, independently of ϕ_0, there exists a stationary measure ρ defined for arbitrary t_i, $i=1,..., n$, by

$$\int d\rho\, F_1(\psi_{t_1})\cdots F_n(\psi_{t_n}) = \lim_{T\to\infty} E_{\phi_0}(F_1(\phi_{T+t_1})\cdots F_n(\phi_{T+t_n})\, e^{\xi_{T+t}})$$

whose truncated correlations satisfy an exponential cluster property. The decay time is smaller than 8. As in Theorem 1, the limit is expressed as a convergent series and is independent of the initial condition.

Since the drift is a polynomial of variables lying in a one-dimensional inner space, one can perform the stochastic integral, using the Ito formula, and get

$$\tfrac{1}{4}\int_0^T d:\phi_s^4: = \tfrac{1}{4}(:\phi_T^4: - :\phi_0^4:) = -\tfrac{1}{2}\int_0^T :\phi_s^4:\, ds + \int_0^T :\phi_s^3:\, dw_s \tag{22}$$

Therefore

$$\xi_T = -\frac{\lambda}{4}\int_0^T :\phi_s^4:\, ds - \frac{\lambda^2}{8}\int_0^T (:\phi_s^3:)^2\, ds - \frac{\lambda}{8}:\phi_T^4: + \frac{\lambda}{8}:\phi_0^4: \tag{23}$$

The expectation value of $F(\phi_t)\, e^{\xi_t}$ is taken with respect to a Gaussian measure μ_C of mean $\phi_t^0 = e^{-t/2}\phi_0$ and covariance

$$C(t,s) = e^{-(1/2)|t-s|}(1 - e^{-\inf(t,s)}) \tag{24}$$

To prove the theorems, we will define an expansion which shows that, for all T, $E_{\phi_0}^T$ is uniformly bounded and satisfied the Cauchy convergence criterion.

Remark. As will be clear from the proofs, the above statements are uniform in the initial condition ϕ_0 as long as this varies in a compact. For the special model equation under study one has exponential convergence in time to the invariant measure uniformly for all values of ϕ_0 due to the fast growth of the drift at infinity. Our result therefore is not the strongest possible for this case. The expansion, however, covers a wide variety of equations for which such a strong uniformity in the initial condition is not expected. The uniformity in F can be presumably extended to a much larger class of functions.

3. THE EXPANSION

The expansion is a kind of cluster expansion defined in such a way that one tries to decouple intervals of time containing the final time T from the other time interval. In this way, one will have to estimate finite-time expressions. The couplings to the whole time interval will be small because of the smallness of the coupling constant.

The expansion will be constructed in such a way that the expressions which are decoupled from the final time T are of the form $E_{\phi_0}(e^{\xi_t})$ for some time $t \in [0, T]$. Therefore, according to the Girsanov theorem, they are equal to 1.

3.1. The Initial Step

To define the initial step one introduces an interpolating parameter s_1, $s_1 \in [0, 1]$. Define the characteristic functions

$$\chi_{k,j}(s) = \chi(s \in]j, k]) \qquad \text{for} \quad k > j \geqslant 1$$

$$\chi_j(s) = \chi_{j,j-1}(s) = \chi(s \in]j-1, j])$$

and

$$\chi_{(k)}(s) = \chi(s \in [0, k])$$

Thus, for example,

$$\chi_{(T)}(s) = \chi_T(s) + \chi_{(T-1)}(s)$$

Then we introduce the interpolating covariance

$$C(s_1)(t, s) = [\chi_T(t)\, C(t, s)\, \chi_T(s) + (1 - \chi_T)(t)\, C(t, s)(1 - \chi_T)(s)](1 - s_1) + C(t, s)\, s_1 \quad (25)$$

and the interpolating "final" condition

$$\phi^4_T(s_1) = \phi^4_T + (1 - s_1)\,\phi^4_{T-1} \tag{26}$$

Replacing in the measure the covariance C by $C(s_1)$ and in ξ_T, ϕ^4_T by $\phi^4_T(s_1)$, thus ξ_T by $\xi_T(s_1)$, we obtain that $E^T_{\phi_0}$ becomes a function $E^T_{\phi_0}(s_1)$ of s_1. Our starting expression is therefore $E^T_{\phi_0}(1)$.

The first step of the expansion is obtained by writing

$$E^T_{\phi_0}(1) = E^T_{\phi_0}(0) + \int_0^1 \frac{d}{ds_1} E^T_{\phi_0}(s_1)\, ds_1 = I_1 E^T_{\phi_0} + D_1 E^T_{\phi_0} \tag{27}$$

where

$$I_1 E^T_{\phi_0} \equiv E^T_{\phi_0}(0) \tag{28}$$

and

$$D_1 E^T_{\phi_0} = \int_0^1 \frac{d}{ds_1} E^T_{\phi_0}(s_1)\, ds_1 \tag{29}$$

3.1.1. The Term I_1. Because of our choice of characteristic functions, $C(0)$ is a direct sum of two terms, $\chi C \chi$ and $(1-\chi)\,C(1-\chi)$. Thus the measure factorizes,

$$d\mu_C = d\mu_{\chi C \chi} \cdot d\mu_{(1-\chi)C(1-\chi)}$$

Correspondingly, one writes

$$\xi_T = \xi_{T,T-1} + \xi_{T-1}$$

where

$$\xi_{T,T-1} = -\frac{\lambda}{4}\int_{T-1}^{T} :\phi^4_s:\, ds - \frac{\lambda^2}{8}\int_{T-1}^{T} (:\phi^3_s:)^2\, ds - \frac{\lambda}{8} :\phi^4_T:$$

$$\begin{aligned} \xi_{T-1} &= -\frac{\lambda}{4}\int_0^{T-1} :\phi^4_s:\, ds - \frac{\lambda^2}{8}\int_0^{T-1} (:\phi^3_s:)^2\, ds \\ &\quad -\frac{\lambda}{8} :\phi^4_{T-1}: + \frac{\lambda}{8} :\phi^4_0: \\ &= -\frac{\lambda}{4}\int_0^{T-1} :\phi^3_s:\, dw_s - \frac{\lambda^2}{8}\int_0^{T-1} (:\phi^3_s:)^2\, ds \end{aligned} \tag{30}$$

Since $\mu_{(1-\chi)C(1-\chi)}$ is an Ornstein–Uhlenbeck measure restricted to the time interval $[0, T-1]$, one has

$$I_1 E_{\phi_0}(F(\phi_T)\, e^{\xi_T}) = E_{\phi_0}(e^{\xi_{T-1}})\, E_{\phi_0}(F(\phi_T)\, e^{\xi_{T,T-1}})$$
$$= E_{\phi_0}(F(\phi_T)\, e^{\xi_{T,T-1}}) \tag{31}$$

where the factorization arises from the support property of the characteristic functions.

It is now easy to show that

$$\lim_{T\to\infty} E_{\phi_0}(F(\phi_T)\, e^{\xi_{T,T-1}}) = E^{\mathrm{St}}(F(\phi_1)\, e^{\xi_1}) \tag{32}$$

where E^{St} means the Gaussian expectation with respect to the stationary covariance

$$C_{\mathrm{St}}(t, s) = e^{-|t-s|/2}$$

and

$$\xi_1 = -\frac{\lambda}{4}\int_0^1 :\phi_s^4:\, ds - \frac{\lambda^2}{8}\int_0^1 (:\phi_s^3:)^2\, ds - \frac{\lambda}{8} :\phi_1^4:$$

Equation (32) is very interesting because the expectation is restricted to trajectories over a finite time interval and the corrections arising from the term D_1 are of order λ, as we show in the next subsection.

3.1.2. The Term D_1. The derivative of $E_{\phi_0}^T(s_1)$ gives

$$\int \frac{d}{ds_1} E_{\phi_0}^T(s_1)\, ds_1 = \int \left(\frac{d}{ds_1}\, d\mu_{C(s_1)}\right) F(\phi_T)\, e^{\xi_T}\, ds_1 + \int F(\phi_T)\, \frac{d\xi_T}{ds_1}\, e^{\xi_T}\, d\mu_{C(s_1)}\, ds_1$$
$$= A_1 + B_1 \tag{33}$$

3.1.3. The Term A_1. As is well known (see, for example, ref. 7).

$$\frac{d}{ds}\, d\mu_{C(s)} = \frac{1}{2}\int du\, dv\, \frac{d}{ds}\, C(s)(u, v)\, \frac{\delta}{\delta\phi_u}\frac{\delta}{\delta\phi_v}\, d\mu_{C(s)}$$

and

$$\frac{dC(s_1)}{ds_1}(u_1, v_1) = (1-\chi_T)(u_1)\, C(u_1, v_1)\, \chi_T(v_1)$$
$$+ \chi_T(u_1)\, C(u_1, v_1)(1-\chi_T)(v_1)$$

which is symmetrical. Thus,

$$\begin{aligned} A_1 &= \int \cdots \int \chi_T(u_1)\, C(u_1, v_1)(1-\chi_T)(v_1) \\ &\quad \times \frac{\delta}{\delta\phi_{u_1}} \left(F(\phi_T) \frac{\delta\xi_T}{\delta\phi_{v_1}} e^{\xi_T} \right) d\mu_{C(s_1)}\, du_1\, dv_1\, ds_1 \\ &= \sum_{j=0}^{T-2} \int \cdots \int \chi_T(u_1)\, C(u_1, v_1)\, \chi_{j+1}(v_1) \\ &\quad \times \frac{\delta}{\delta\phi_{u_1}} \left(F(\phi_T) \frac{\delta\xi_T}{\delta\phi_{v_1}} e^{\xi_T} \right) d\mu_{C(s_1)}\, du_1\, dv_1\, ds_1 \\ &= \sum_{j=0}^{T-2} A_1(j) \end{aligned} \tag{34}$$

We shall refer to $[j, T]$ as the support of $A_1(j)$.

3.1.4. The Term B_1. We have

$$B_1 = \frac{\lambda}{8} \int F(\phi_T) :\phi_{T-1}^4: e^{\xi_T}\, d\mu_{C(s_1)}\, ds_1 \tag{35}$$

In conclusion, the term D_1 is of order λ uniformly in T (as will follow from the ensuing discussion).

3.2. The Next Step

The expansion ends for the I_1 term. The A_1 and the B_1 terms are expanded further.

3.2.1. The Next Step of the Expansion for the A_1 Term. The A_1 term is a sum of terms $A_1(j)$, $j = 0,..., T-2$. We describe the expansion for $A_1(j_1)$ for some j_1, $0 \leqslant j_1 < T-1$. As for the first step, we introduce an interpolating parameter s_2 and define an interpolating covariance $C(s_1, s_2)$ which will help us to test to what extent the interval $]j_1, T]$, the support of $A(j_1)$, is connected by a covariance to its complementary set in $[0, T]$,

$$\begin{aligned} C(s_1, s_2)(t, s) &= [\chi_{T,j_1}(t)\, C(s_1)(t, s)\, \chi_{T,j_1}(s) \\ &\quad + (1-\chi_{T,j_1})(t)\, C(s_1)(t, s)(1-\chi_{T,j_1})(s)](1-s_2) \\ &\quad + C(s_1)(t, s)\, s_2 \end{aligned} \tag{36}$$

We also introduce

$$\phi_T^4(s_1, s_2) = \phi_T^4 + (1 - s_1)\,\phi_{T-1}^4 + (1 - s_2)\,\phi_{j_1}^4$$

and therefore an interpolating $\xi_T(s_1, s_2)$; $A_1(j_1)$ is the value at $s_2 = 1$ of $A_1(j_1)(s_2)$ and we rewrite it as

$$A_1(j_1) = I_2 A_1(j_1) + D_2 A_1(j_1)$$

3.2.2. The Term $I_2 A_1(j_1)$. The $I_2 A_1(j_1)$ term factorizes as

$$I_2 A_1(j_1) = \int E_{\phi_0}(e^{\xi_{j_1}})\, E_{\phi_0}(P_2(\phi)\, e^{\xi_{T,j_1}(s_1, 0)})\, ds_1 \tag{37}$$

where

$$\begin{aligned} P_2(\phi) = & \int \cdots \int \chi_T(u_1)\, C(u_1, v_1)\, \chi_{j_1+1}(v_1) \\ & \times \left(\frac{\delta}{\delta\phi_{u_1}} F(\phi_T)\right) \frac{\delta \xi_T}{\delta \phi_{v_1}}\, e^{\xi_{T,j_1}}\, du_1\, dv_1 \\ & + \int \cdots \int \chi_T(u_1)\, C(u_1, v_1)\, \chi_{j_1+1}(v_1)\, F(\phi_T) \\ & \times \frac{\delta \xi_T}{\delta \phi_{v_1}} \frac{\delta \xi_T}{\delta \phi_{u_1}}\, e^{\xi_{T,j_1}}\, du_1\, dv_1 \end{aligned} \tag{38}$$

and

$$\xi_T(s_1, 0) = \xi_{T,j_1}(s_1, 0) + \xi_{j_1}$$

with

$$\begin{aligned} \xi_{T,j_1}(s_1, 0) = & -\frac{\lambda}{4} \int_{j_1}^{T} :\phi_s^4:\, ds - \frac{\lambda^2}{8} \int_{j_1}^{T} (:\phi_s^3:)^2\, ds \\ & -\frac{\lambda}{8}(:\phi_T^4: + (1 - s_1) :\phi_{T-1}^4:) \\ \xi_{j_1} = & -\frac{\lambda}{4} \int_0^{j_1} :\phi_s^4:\, ds - \frac{\lambda^2}{8} \int_0^{j_1} (:\phi_s^3:)^2\, ds \\ & -\frac{\lambda}{8} :\phi_{j_1}^4: + \frac{\lambda}{4} :\phi_0^4: \\ = & -\frac{\lambda}{2} \int_0^{j_1} :\phi_s^3:\, dw_s - \frac{\lambda^2}{8} \int_0^{j_1} (:\phi_s^3:)^2\, ds \end{aligned} \tag{39}$$

In the last step, we have used the fact that, in $[0, j_1]$, ϕ_s is an Ornstein–Uhlenbeck process.

Thus by the Girsanov formula

$$I_2 A_1(j_1) = \int E_{\phi_0}(P_2(\phi)\, e^{\xi_{T,j_1}})\, ds_1 \tag{40}$$

3.2.3. The Term $D_2 A_1(j_1)$. This is a sum of terms

$$D_2 A_1(j_1) = \sum_{j_2=0}^{j_1-1} A_{1,2}(j_1, j_2) \tag{41}$$

where

$$\begin{aligned} &A_{1,2}(j_1, j_2) \\ &\quad = \int d\mu_{C(s_1,s_2)} \left\{ \iint du_2\, dv_2\, \chi_{T,j_1}(u_2)\, C(u_2, v_2)(s_1) \right. \\ &\quad \times \chi_{j_2+1}(v_2) \frac{\delta}{\delta\phi_{u_2}} \frac{\delta}{\delta\phi_{v_2}} \left[\iint du_1\, dv_1\, \chi_T(u_1)\, C(u_1, v_1)\, \chi_{j_1+1}(v_1) \right. \\ &\quad \left. \left. \times \frac{\delta}{\delta\phi_{u_1}} \left(F(\phi_T) \frac{\delta\xi_T}{\delta\phi_{v_1}} e^{\xi_{T(s_1,s_2)}} \right) \right] \right\} ds_1\, ds_2 \end{aligned} \tag{42}$$

The interval $[j_2, T]$ is called the support of $A_{1,2}(j_1, j_2)$.

3.2.4. The Next Step of the Expansion for the B_1 Term. This term which resembles the initial term, with the function to evaluate $F(\phi_T)$ replaced by $\lambda F(\phi_T)\ {:}\phi_{T-1}^4{:}$, is smaller than the initial term since it has a λ in front of it. We will consider it as a term with support $[T-2, T]$. Therefore we will apply to it the initial step of the expansion, with χ_T replaced by $\chi_{T,T-2}$ and

$$\phi_T^4(s_1, s_2) = \phi_T^4 + (1 - s_1)\, \phi_{T-1}^4 + (1 - s_2)\, \phi_{T-2}^4$$

The formulas are then the same as for $A_1(T-2)$.

3.3. The Generic Step

At the end of the $(k-1)$th step, one has an interpolated covariance $C(s_1,..., s_{k-1})$. At each step an s_l is introduced and the corresponding operation D_l leads to a sum of terms labeled by an integer j_l if we are dealing with an A term, or to one term if it is a B. To simplify the notation, we introduce $\tilde{D}(j_l)$, which stands for both A and B, with $j_l < j_{l-1}$ if A is chosen and $j_l = j_{l-1} - 1$ if B is chosen. Thus, for a given term made of D's, to $s_1, ..., s_{k-1}$ is associated a multiple sum over $j_1, ..., j_{k-1}$ with $T > j_1 > \cdots > j_{k-1}$.

One now introduces the new parameter s_k.

By definition,

$$\begin{aligned} &C(s_1, ..., s_k)(t, s) \\ &\quad = [\chi_{T, j_{k-1}}(t)\, C(s_1, ..., s_{k-1})(t, s)\, \chi_{T, j_{k-1}}(s) \\ &\qquad + (1 - \chi_{T, j_{k-1}})(t)\, C(s_1, ..., s_{k-1})(t, s)(1 - \chi_{T, j_{k-1}}(s)](1 - s_k) \\ &\qquad + C(s_1, ..., s_{k-1})(t, s)\, s_k \end{aligned} \tag{43}$$

and

$$\phi_T^4(s_1, ..., s_k) = \phi_T^4 + \sum_1^k (1 - s_l)\, \phi_{j_{l-1}}^4$$

with $j_0 = T - 1$.

One introduces this new dependence in the last nonfactorized term of the expansion

$$\begin{aligned} D_{k-1} \cdots D_1 E_{\phi_0}^T &= D_{k-1} \cdots D_1 E_{\phi_0}^T |_{s_k = 0} + \int_0^1 ds_k \frac{d}{ds_k} D_{k-1} \cdots D_1 E_{\phi_0}^T \\ &= (I_k + D_k)\, D_{k-1} \cdots D_1 E_{\phi_0}^T \end{aligned} \tag{44}$$

Formally, the expansion at the kth step has the form

$$E_{\phi_0}^T = I_1 E_{\phi_0}^T + I_2 D_1 E_{\phi_0}^T + \cdots + I_k D_{k-1} \cdots D_1 E_{\phi_0}^T + D_k \cdots D_1 E_{\phi_0}^T \tag{45}$$

where

$$\begin{aligned} &I_k D_{k-1} \cdots D_1 E_{\phi_0}^T \\ &\quad = \int_0^1 ds_1 \cdots \int_0^1 ds_{k-1} \frac{d}{ds_{k-1}} \cdots \frac{d}{ds_1} \int d\mu_{C(s_1, ..., s_k)}\, F(\phi_T)\, e^{\xi_T(s_1, ..., s_k)} |_{s_k = 0} \\ &\quad = \sum_{T > j_1 > \cdots > j_{k-1}} \sum_{\tilde{D} = A \text{ or } B} I_k \tilde{D}_{k-1}(j_{k-1}) \cdots \tilde{D}_1(j_1)\, E_{\phi_0}^T \end{aligned} \tag{46}$$

The exponent, as far as the time interval $[j_{k-1}, T]$ is concerned, is, for a given choice of $j_1, ..., j_{k-1}$, reduced to

$$\xi_{T,j_{k-1}}(s_1,..., s_{k-1}) = -\frac{\lambda}{4}\int_{j_{k-1}}^{T} :\phi_s^4: ds - \frac{\lambda^2}{8}\int_{j_{k-1}}^{T} (:\phi_s^3:)^2 ds$$
$$- \frac{\lambda}{8} :\phi_T^4(s_1,..., s_{k-1}): \tag{47}$$

which corresponds to a localization in $\chi_{j_{k-1}+1}$ at the $(k-1)$th step. The interval $]j_{k-1}, T]$ will be called the support of $I_k\tilde{D}_{k-1}(j_{k-1})\cdots \tilde{D}_1(j_1)\,E^T_{\phi_0}$.

Remark that since each step localizes the support at least one unit interval to the left, then $j_{k-1} \leqslant T-k$.

Similarly,

$$D_kD_{k-1}\cdots D_1E^T_{\phi_0}$$
$$= \int_0^1 ds_1\cdots\int_0^1 ds_k \frac{d}{ds_k}\cdots\frac{d}{ds_1}\int d\mu_{C(s_1,...,s_k)}\,F(\phi_T)\,e^{\xi_T(s_1,...,s_k)} \tag{48}$$

with

$$\xi_T(s_1,..., s_k) = -\frac{\lambda}{4}\int_0^T :\phi_s^4: ds - \frac{\lambda^2}{8}\int_0^T (:\phi_s^3:)^2 ds$$
$$- \frac{\lambda}{8} :\phi_T^4(s_1,..., s_{k-1}): + \frac{\lambda}{8} :\phi_0^4: \tag{49}$$

Terms like $D_nD_{n-1}\cdots D_1E^T_{\phi_0}$ can be written as sums, over the localization indices $j_1,..., j_n$ with $j_m < j_l$ if $m < l$, of $\tilde{D}_n(j_n)\,\tilde{D}_{n-1}(j_{n-1})\cdots\tilde{D}_1(j_1)\,E^T_{\phi_0}$.

The expansion ends when $\chi_{T,j_k} = \chi_{(T)}$, i.e., $j_k = 0$, that is, at most after T steps.

We will show in the next section that for λ small enough there exist two constants $K>0$ and $\mu<1$ such that

$$|I_kD_{k-1}\cdots D_1E^T_{\phi_0}| < K\mu^{k-1} \tag{50}$$

uniformly in T.

4. THE BOUNDS

Our first task is the Wick bound, i.e., a bound on e^{ξ}. After giving the lemmas useful to bound the various terms produced by the expansion, we obtain an estimate on the number of terms generated by the expansion and finally prove the announced result.

4.1. The Wick Bound

According to the form of the generic term of the expansion, we need to bound $e^{\xi_{T,j}}$. We are therefore looking at an upper bound for $\xi_{T,j}$. Since this term is made of Wick polynomials, we use

$$:\phi_s^4: \geqslant -6$$

and

$$:\phi_T^4(s_1,..., s_k): \geqslant -6\left[1 + \sum_{m=1}^{k} (1 - s_m)\right] \geqslant -6(k+1)$$

Thus

$$e^{\xi_{T,j_k}} \leqslant e^{(6\lambda/8)(k+1)+(6\lambda/4)(T-j_k)} \tag{51}$$

where we used the positivity of $(:\phi^3:)^2$.

4.2. The Basic Lemmas

We first prove a simple lemma for Gaussian variables ϕ_i of mean 0 and covariance C such that

$$C(s, t) = C(t, s) \leqslant \alpha \frac{1}{(1+\beta\,|s-t|)^{1+\varepsilon}}$$

We denote by $\langle \cdot \rangle$ the expectation with respect to this measure.

Lemma 1. Let $P_j(x)$, $j = 1,..., n$, be n polynomials and let $s_i \in\,]l(i)-1, l(i)]$, $l(i)$ being positive integers. Moreover, suppose $l(1) < \cdots < l(n)$, that is, the s_i belong to nonoverlapping unit intervals; then there exists a constant K such that

$$\left\langle \prod_1^n P_j(\phi_{s_j}) \right\rangle \leqslant (NN!^{1/2}K)^n \tag{52}$$

with $N = \sup_j d^\circ(P_j)$, where $d^\circ(P_j)$ is the degree of P_j. The constant K depends on α, β, ε, and $a = \sup_{ij} |a_{ij}|$, the a_{ij} being the coefficients of the polynomial $P_j(x)$.

Proof. We give the proof for monomials; thus $P_j(x) = x^{n_j}$ and $N = \sup_j n_j$.

Since the measure is Gaussian, the expression is different from 0 only if $\sum_i n_i$ is even, i.e., $2L=\sum_j n_j$. In this case

$$\left\langle \prod_1^n \phi_{s_j}^{n_j} \right\rangle = \sum_I d_I$$

where the sum runs over all distinct partitions of $\{s_1,..., s_1, s_2,..., s_2,..., s_n,..., s_n\}$, s_j appearing n_j times, into L pairs, and d_I is the product of the L covariances whose endpoints are the two elements of each pair (in field theory language, this is nothing else than the Wick theorem, which says that the Gaussian integration of a product of fields is the sum over all possible products of expectations of pairs of fields).

To get our result we use the method of combinatorial factors, i.e., we introduce, for all I, strictly positive numbers c_I and apply the bound

$$\sum_I d_I \leqslant \left(\sum_I c_I^{-1}\right) \sup_I c_I d_I \tag{53}$$

Obviously, to be interesting, this bound requires that the combinatorial factors c_I have something to do with the behavior of d_I with respect to the partition. This point will be illustrated in the sequel. Clearly this formula applies also in the case the partitions run over denumerable sets, provided

$$\sum_I c_I^{-1} < \infty \tag{54}$$

The following remark, which is part of the power of the method, will be used extensively:

If the sum over the indices I can be rearranged in terms of two sums over sets of indices I_1 and I_2, $I=I_1 \cup I_2$ (for example, corresponding to conditional labeling), so that

$$\sum_I d_I = \sum_{I_1} \left(\sum_{I_2} d_{I_1, I_2} \right)$$

then the combinatorial factor c_I related to the sum over I can be written as a product of combinatorial factors $c_I = c_{I_1} c_{I_2}$ related to each of the sums.

Since $\sum_I c_I^{-1} < \infty$ and the estimate of the r.h.s. of (53) is independent of the scale of c_I, we can replace c_I by $c_I \sum_J c_J^{-1}$ and decide from now on that the combinatorial factors are such that

$$\sum_I c_I^{-1} \leqslant 1$$

We now explain how, in the case of the lemma, one can define the combinatorial factors c_I. We will do it in a constructive way.

A way to understand the rules of Gaussian calculus is by doing integration by part with respect to the Gaussian measure:

$$\langle \phi_s F(\phi) \rangle = \int C(s, t) \left\langle \frac{\delta F(\phi)}{\delta \phi_t} \right\rangle dt \tag{55}$$

where F is some polynomial functional of the variables ϕ_u. In this formula two variables ϕ_s and ϕ_t have been replaced by a covariance $C(s, t)$. Thus what remains is F with one variable ϕ_t less, a fact which is mathematically described by the functional derivative of F with respect to ϕ_t. We will say that ϕ_s has contracted to ϕ_t or equivalently that ϕ_t has contracted to ϕ_s, since the covariance is a symmetrical function of the endpoints. Remark that when F is a polynomial, repeated applications of formula (55) lead to a computation of the expectation value in term of products of covariances. This is the method we will use now to give a bound on the left-hand side of formula (52).

The contraction process, i.e., picking a variable, then associating to it any other variable and replacing these two variables by their covariance, repeated up to the complete exhaustion of variables, is an explicit way of describing the result. Obviously the result is independent of the choice of the initial variables. This means that one can choose the order in which to pick the first variable in each pair of contracted variables. This order will be the order of increasing times and we will start from one of the n_1 variables localized at s_1; then when all the n_1 variables have been contracted, we choose one of the n_2 variables, which now can only contract to variables localized at s_j with $j \geqslant 2$, and so on. Therefore with this choice of rule for performing the contractions, a set I is described inductively as follows (Fig. 1): given one of the variables ϕ_{s_1}, $s_1 \in]l(1)-1, l(1)]$, set $i_1 = l(1)$ and choose the time interval $[j_1 - 1, j_1]$ in which the variable lies which will be contracted with ϕ_{s_1}. Since $j_1 \geqslant i_1$, $j_1 = l(\tau(1))$ for some integer $1 \leqslant \tau(1) \leqslant n$; then choose which one of the $n_{\tau(1)}$ variables $\phi_{s_{\tau(1)}}$ will be contracted [if $\tau(1) = 1$, there remain only $n_1 - 1$ variables to be chosen]. This will be the first contraction. Take then another variable localized in $]i_2 - 1, i_2]$ [$i_2 = l(2)$ either if $n_1 = 1$ or if $n_1 = 2$ and $\tau(1) = 1$; $i_2 = l(1)$ otherwise] and repeat the procedure; if there are no more variables in $]l(1)-1, l(1)]$, then take a variable in $]l(2)-1, l(2)]$ and do as before,.... Generically, if the initial variable is in $]i_m - 1, i_m]$, it can contract to any variable in one of the intervals $]l(p)-1, l(p)]$,..., $]l(n)-1, l(n)]$, with p such that $l(p) \geqslant i_m$, and once the interval has been chosen, say $l(\tau(m))$, $p \leqslant \tau(m) \leqslant n$,

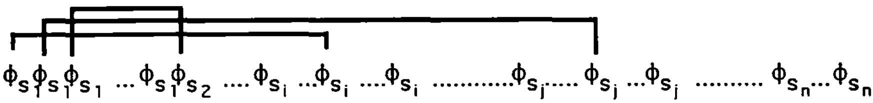

Fig. 1. The contraction procedure.

there are at most $\eta_{\tau(m)}$ variables to choose. This means that fixing a set I is equivalent to choosing j_1, $j_1 \geqslant i_1$, then j_2, $j_2 \geqslant i_2$,..., where

$$\{l(1),..., l(1), l(2),..., l(2),..., l(n),..., l(n)\} = \{i_1,..., i_L\} \cup \{j_1,..., j_L\}$$

i.e., fixing the pairings of the unit time intervals linked by contraction and then choosing in each contraction process which variable in a given unit time interval is effectively contracted. We call I_1 the set of all pairings $\{(i_1, j_1),..., (i_L, j_L)\}$. Then I_1 being fixed, we have to sum over all possible choices of variables in $]\tau(1)-1, \tau(1)],...,]\ \tau(L)-1, \tau(L)]$. We call I_2 the sets of indices labeling these choices.

Thus

$$\left\langle \prod_1^n \phi_{s_j}^{n_j} \right\rangle = \sum_I d_I = \sum_{I_1} \left(\sum_{I_2} C(s_{\sigma(1)}, s_{\tau(1)}) \cdots C(s_{\sigma(L)}, s_{\tau(L)}) \right) \tag{56}$$

where

$$\{s_{\sigma(1)}, s_{\tau(1)},..., s_{\sigma(L)}, s_{\tau(L)}\} = \{s_1,..., s_1,..., s_n\}$$

and $s_{\sigma(k)} \in\ ]i_k - 1, i_k]$, $s_{\tau(k)} \in\ ]j_k - 1, j_k]$ for $k = 1,..., L$.

It results that the combinatorial factor for the sum over I is given by the product of two combinatorial factors, one is c_{I_1} and is related to fixing the time intervals, say $\chi_{\tau(m)}$, $j_m = l(\tau(m))$, in which lies the mth contracted variable, $m = 1,..., L$, and the other one is c_{I_2} and is related to which one of the $n_{\tau(m)}$ variables will be contracted.

We choose

$$c_{I_1} = K_1(\varepsilon)^L \prod_m (1 + |j_m - i_m|)^{1+\varepsilon}$$

where $K_1(\varepsilon)$ is such that

$$\sum_{I_1} \prod_m (1 + |j_m - i_m|)^{-(1+\varepsilon)} = \prod_{m=1}^{L} \left(\sum_{j_m \geqslant i_m} (1 + |j_m - i_m|)^{-(1+\varepsilon)} \right) \leqslant K_1(\varepsilon)^L$$

Moreover, $c_{I_2} = N^L$, since, I_1 being fixed,

$$\sum_{I_2} 1 \leqslant \prod_{m=1}^{L} n_{\tau(m)} \leqslant N^L$$

Thus we can take

$$c_I = K_1(\varepsilon)^L N^L \prod_m (1 + |j_m - i_m|)^{1+\varepsilon} \tag{57}$$

Now, since

$$C(s_{\sigma(k)}, s_{\tau(k)}) \leqslant \alpha \left(\frac{1+\beta}{1+\beta\, |j_k - i_k|} \right)^{1+\varepsilon}$$

$$\begin{aligned}
\left\langle \prod_1^n \phi_{s_j}^{n_j} \right\rangle &= \sum_I d_I \leqslant \sup_{I_1, I_2} c_{I_1} c_{I_2} d_I \\
&\leqslant (N\alpha K_1)^L \sup \prod_m \frac{(1 + |j_m - i_m|)^{1+\varepsilon}}{(1+\beta\, |j_m - i_m|)^{1+\varepsilon}} \\
&\leqslant (N\alpha K_1 K_2)^L
\end{aligned} \tag{58}$$

where $K_1 = K_1(\varepsilon)\ (1+\beta)^{1+\varepsilon}$ and $K_2 = \max(2, \beta^{-1})^{1+\varepsilon}$, since

$$\frac{1 + |j_m - i_m|}{1 + \beta\, |j_m - i_m|} \leqslant \max(2, \beta^{-1})$$

Thus, since $2L \leqslant nN$, one gets the result for monomials

$$\left\langle \prod_1^n \phi_{s_j}^{n_j} \right\rangle \leqslant (N!^{1/2} K)^n \tag{59}$$

with $K = (\alpha K_1 K_2)^{N/2}$. From this bound, if the monomials are replaced by polynomials, the inequality of the lemma follows easily by considering that the number of monomials in each P_j is bounded by $N+1$.

We will use intensively in the sequel a straightforward extension of the above bound. The bound is still true if one replaces everywhere

$$\phi_{s_k}^{n_k} \to \phi_{l(k)}^{(n_k)} \equiv \prod_{r=1}^{n_k} \phi_{s_{k_r}} \tag{60}$$

with $s_{k_r} \in \,]l(k)-1,\ l(k)]$, for $k = 1, \ldots$

Remark. The bound of the lemma shows the characteristic $N!^{1/2}$ behavior expected from the contractions of N variables located in the same time interval.

We now prove another lemma, which makes the dependence of the bound on the local number of variables more precise. Under the same conditions as in Lemma 1, one has the following.

Lemma 2. The following condition holds:

$$\left\langle \prod_{1}^{n} \phi_{s_j}^{n_j} \right\rangle \leqslant \prod_{j} (n_j!)^{1/2} K^n \tag{61}$$

Proof. The proof is identical as the proof of Lemma 1 except for the combinatorial factors associated with the choice of variables at a given site.

We order the variables decreasingly according to n_j. Thus, let $j(1),..., j(n)$ be a reordering of the first n integers such that $n_{j(1)} \geqslant n_{j(2)} \cdots \geqslant n_{j(n)}$. Then we start the contractions with the variables labeled $j(1)$, then with the variables labeled $j(2),...$. A variable $\phi_{j(1)}$ can contract to another variable $\phi_{j(m)}$. Since the number of such variables is at most $n_{j(m)}$, one can take n_m as combinatorial factor. Now because of the ordering, we can use $n_{j(m)} \leqslant (n_{j(1)})^{1/2}\ (n_{j(m)})^{1/2}$ to replace the initial combinatorial factor by a more symmetrical one: the square root of the product of the number of contracting variables by the number of contracted variables of the required type. Repeating this analysis for the other contractions, we get as a combinatorial factor, instead of $N^{N/2}$, $\prod_j n_j^{n_j/2}$. Then, using the fact that $p^{p/2} \leqslant C^p p!^{1/2}$, one gets the announced bound with some new K independent of the number of variables.

This type of lemma can be found, for example, in ref. 10, Chapter 2, §3. We have given an independent proof to make the paper as self-contained as possible.

4.3. Bound of a Generic Term of the Expansion

A generic term of the expansion is of the form $I_k D_{k-1} \cdots D_1 E_{\phi_0}^T$, which is in turn a sum of terms. Since the expectation i linear in F, one has, prior to any expansion, at most $d^\circ F + 1$ contributions corresponding to the different monomials appearing in F. The other sums, those generated by the expansion, are either related to the choice of the time localizations of the D's, i.e., of a sequence of supports indexed by $j_1 > \cdots > j_{k-1}$, or to the choice to be made among identical objects.

Let us first treat this last case. Since $D = A + B$, there are 2^{k-1} distinct ways to write an ordered sequence of $k-1$ products of A or B. This number is the combinatorial factor. The A terms are made of derivatives. Each derivative can act either on the exponent or on already produced terms, i.e., it gives rise to two sets of terms. This gives a combinatorial factor 2 per derivative. Since the terms to derive, new ones or already derived ones, are local polynomials of degree at most $p' = d°F$ if F is differentiated or $p = 6$ for the other case, we have, per functional derivative, a corresponding combinatorial factor p' or $p \times p = p^2$. In the latter case one p is to choose which monomial is derived and the other one to choose which variable is derived (the monomials are at most of degree p). This gives, per step, i.e., two derivatives, a final combinatorial factor $K_1 = (2p^2)^2$ except if it is F which is hit by one of the derivatives, in which case one has to replace K_1 by $K'_1 = 4p'p^2$. We then have to estimate sums over the time localizations of products of monomials.

The sum over time localizations is a relevant question only for the A terms, since the B terms generate localized variables. For an A term one is led, say, as the mth step, to derive variables at a time u_m contained in a known time interval $[j_{m-1}, T]$, the support at the $(m-1)$th step, and variables at a time v_m contained in its complementary set in $[0, T]$. To define inductively the steps of the expansion, it is necessary to fix the unit time interval in which $[j_m, j_m+1]$ is localized v_m, the A term being written as a sum over j_m such that $0 \leqslant j_m < j_{m-1}$. The new support at step m is then defined as $[j_m, T]$ (Fig. 2). To make the bounds on the various terms of the expansion easier, it will also be convenient to restrict the integration of the time u_m to unit time intervals $[i_m, i_m+1]$, with $i_m \geqslant j_{m-1}$. This means that each A term is now the sum of expressions labeled by pairs of numbers $\{(i_1, j_1),..., (i_m, j_m),...\}$ where $i_l \geqslant j_l > j_{l+1} > 0$, $l = 1,..., k-1$, and $i_1 = T-1$. We will have to sum over all allowed values of these pairs of variables, a task that will be done easily because, by the way the expansion is defined, there are decreasing propagators connecting the unit time intervals defined by each pair of variables.

Remark that if at the mth step a B term is chosen, then there is no summation over i_m and $j_m = j_{m-1} - 1$.

An expression like $I_k D_{k-1} \cdots D_1 E^T_{\phi_0}$ can be written, with an obvious generalization of the notation introduced in Section 3.3,

$$I_k D_{k-1} \cdots D_1 E^T_{\phi_0} = \sum_{i_1, j_1} \cdots \sum_{i_{k-1}, j_{k-1}} I_k \tilde{D}_{k-1}(i_{k-1}, j_{k-1}) \cdots \tilde{D}_1(i_1, j_1)\, E^T_{\phi_0}$$

Each time $\tilde{D}_m$ is an A term, it contributes to the r.h.s. of the above equality a covariance $C(u_m, v_m)$ times a polynomial in ϕ_{u_m} and a polynomial in ϕ_{v_m}, which are functional derivatives of ξ_T or F or of their derivatives; each time $\tilde{D}_m$ is a B term, it contributes by $:\phi^4_{j_{m-1}}:$.

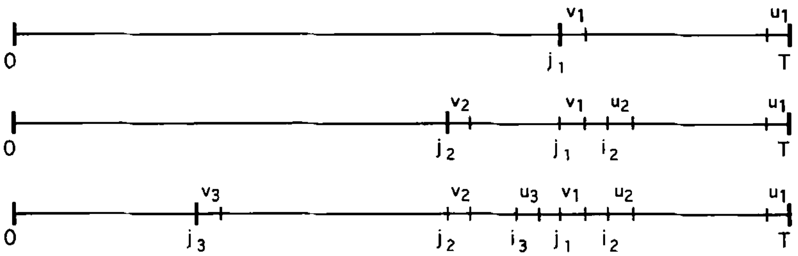

Fig. 2. The supports and the locations of the endpoints of the covariances for the first three steps.

Each element of the r.h.s. of the above equality can be written as the expectation of e^{ξ_T} times the product of polynomials of ϕ_{u_m} and ϕ_{v_m} (monomials from functional derivatives of ξ_T or F) for $u_m \in [i_m, i_m + 1]$ and $v_m \in [j_m, j_m + 1]$, $1 \leqslant m \leqslant k-1$, times the product of covariances $C(u_m, v_m)$ or B terms.

Thus

$$I_k \tilde{D}_{k-1}(i_{k-1}, j_{k-1}) \cdots \tilde{D}_1(i_1, j_1)\, E^T_{\phi_0} \leqslant C_F (K_1 K_2)^{(k-1)} \sup_{\{P\}} |I_{\{P\}}(i_1, j_1, \ldots, i_{k-1}, j_{k-1})|$$

where

$$C_F = (d^\circ F + 1)(K'_1)^{d^\circ F} K'_2$$

K'_2 the supremum of the coefficients of F, while K_2 is the supremum of the coefficients of ξ_T. Now, $I_{\{P\}}(i_1, j_1, \ldots, i_{k-1}, j_{k-1})$ is generically of the form

$$\begin{aligned} &I_{\{P\}}(i_1, j_1, \ldots, i_{k-1}, j_{k-1}) \\ &\quad = \int d\mu_C\, e^{\xi_T} \left\{ \iint \prod_r P_r(\phi_{x_r}) \prod \delta(\cdot) \prod_m C(u_m, v_m)\, \chi_{j_m+1}(u_m) \right. \\ &\qquad \left. \times \chi_{i_m+1}(v_m)\, du_m\, dv_m \right\} \end{aligned} \tag{62}$$

where:

- P_r are some monomials obtained from functional derivatives of F or ξ (one of the P is a monomial from F or from its functional derivatives) whose arguments are variables x_r localized at elements of $J_{k-1} = \{u_1, v_1, \ldots, u_{k-1}, v_{k-1}\}$, i.e., $x_r \in J_{k-1}$.

- The δ-functions have arguments which identify two variables of J_{k-1} (this arises from multiple functional derivatives acting on the same local monomial of the variable, i.e., u_m can be identical with $v_{m-1}, u_{m-1},..., v_1, u_1$).
- The product over covariances runs over all the values of m corresponding to the choice of a A term.

This means that the time integrations in (62) can be reduced using the delta functions to a subset of distinct u and v variables (the B term being omitted since does not carry integrations). Moreover, since at the end we will apply Lemma 1, we can regroup all monomials (of degree less than 6, since ξ is at most of degree 6, except for one of them related to F) whose variables are localized in the same unit time interval.

Since some of the integration intervals may coincide, we rewrite (62) as

$$\int d\mu_C \left\{ \int \cdots \int \prod_m C(u_m, v_m) \prod_{l=j_{k-1}+1}^{T} R_l \, dx_1 \, dx_2 \cdots \right\} e^{\xi_T}$$

where R_l is the product of all the monomials P_n such that the times x_n are localized in $]l-1, l]$. We set $d^\circ R_l = n_l$. Since there are at most $2(k-1)$ functional derivatives and since the polynomials in ξ_T are at most of degree 6, one has the bound

$$\sum_l n_l \leqslant 10(k-1) + d^\circ F \leqslant 10k + d^\circ F$$

the last term of the r.h.s. resulting from the fact that one of the P monomials comes from F or its functional derivatives.

One then applies the Schwarz lemma with respect to the Ornstein–Uhlenbeck measure to separate $\prod R_l$ from the exponential. The term containing the exponential will be estimated using the Wick bound; the other term is then no more than a Gaussian expectation of a product of monomials of variables ϕ_s. To stick more to the usual Gaussian calculus, let us write them

$$\phi_s = \phi_0 e^{-s/2} + \hat{\phi}_s \tag{63}$$

where $\hat{\phi}_s$ is a Gaussian variable of mean 0 and covariance $C(s, t)$.

To simplify our further discussion, we will do the estimate as if all variables, say in R_l, are localized at the same time s_l, $l-1 \leqslant s_l \leqslant l$. Therefore we are led to estimate expressions like

$$I = \left(\int d\mu_C \prod_l \phi_{s_l}^{2n_l} \right)^{1/2}$$

with the above conventions. Then, using (63) and Lemma 2, we have

$$I=\left(\int d\mu_C \prod_l \left\{\sum_{p_l} \phi_0^{q_l} e^{-(1/2)s_l q_l} \hat{\phi}_{s_l}^{p_l} \binom{n_l}{p_l}\right\}\right)^{1/2}$$
$$\leqslant 2^{(\Sigma_l n_l)} \left(\sup_{p_1, q_1, p_2, q_2, \ldots} \prod_l \phi_0^{q_l} e^{-(1/2)s_l q_l} \int d\mu_C \prod_l \hat{\phi}_{s_l}^{p_l}\right)^{1/2}$$
$$\leqslant (2K)^{(\Sigma_l n_l)} \left(\sup_{p_1, q_1, p_2, q_2, \ldots} \prod_l \phi_0^{q_l} e^{-(1/2)s_l q_l} \prod_l (p_l!)^{1/2}\right)^{1/2} \tag{64}$$

where the supremum is taken over all positive integers p_1, $q_1, p_2, \ldots$ such that $p_l + q_l = 2n_l$, $l = 1, \ldots$, and the last bound results from the application of Lemma 2 using the fact that for any $\varepsilon > 0$ (we can choose here $\varepsilon = 1$)

$$C(s, t) \leqslant e^{-|s-t|/2} \leqslant C(\varepsilon) \frac{1}{(1 + \frac{1}{2}|s-t|)^{1+\varepsilon}}$$

for some $C(\varepsilon) > 0$.

A special role will be played by the covariances explicitly appearing in formula (62). We factorize each of these propagators into four parts

$$C(s, t) = C(s-t) = K(s-t)^4 \tag{65}$$

Obviously $K(s-t) \leqslant e^{-|s-t|/8}$.

We will use these four parts to get rid of the factorials, to control the Wick bound, to sum over all localization choices, and finally to exhibit an overall exponential decrease in the length of the support of the expansion term.

We first show how to get rid of the factorials produced by an accumulation of functional derivatives in a given unit time interval. In such a time interval, the accumulation of variables can arise from repeated derivation of the exponential. This occurs each time $\tilde{D}$ is an A term, that is, when the measure is derived and one of the functional derivatives acts on the exponential. But then by construction the other functional derivative is localized further and further from the given time interval, since each time there is a new step the support moves at least one step to the left. We then use the decrease of the covariance linking two localizations to control the factorials. More precisely, if there are n_i variables in the time interval $[i-1, i]$ and n_i is a large number, this means one can associate to this time interval a fraction of the decrease of the covariances obtained by derivation of the measure

$$\prod_{\alpha, \beta \in I_i} C(u_\alpha - v_\beta)$$

where

$$I_i = \{\alpha, \beta \mid u_\alpha \in [i-1, i], v_\beta \in [0, i-1]\}$$

Since each derivative acting on the exponential can produce at most monomials of degree 5 and since there can be, by construction, only one time v in each unit time interval, the smallest value that can take $\sup_{\alpha,\beta \in I_i} |u_\alpha - v_\beta|$ is bigger than $[n_i/5]-1$. Thus, using (65), provided n_i is large enough, we have

$$\prod_{\alpha,\beta \in I_i} K(u_\alpha - v_\beta) \leqslant \exp\left(-\frac{1}{8} \sum_{p=0}^{[n_i/5]-1} p\right) = \exp\left\{-\frac{1}{16}\left(\left[\frac{n_i}{5}\right]-1\right)\left(\left[\frac{n_i}{5}\right]\right)\right\}$$
$$\leqslant K_3 \exp\left(-\frac{n_i^2}{1000}\right) \tag{66}$$

with K_3 independent of n_i.

We will use this estimate for two tasks. One, as announced, is to get rid of the factorials and the second is to get rid of the dependence with respect to the initial condition ϕ_0.

To get rid of the factorials we use the fact that for any $q>0$, there exists $K_4 = K_4(q)$ such that

$$(n_i!)^q \, e^{-n_i^2/2000} \leqslant K_4^{n_i} \tag{67}$$

One thus sees that in the bound of $I_{\{P\}}(i_1, j_1,..., i_{k-1}, j_{k-1})$, the local factorials can be replaced by powers.

We want to control the ϕ_0 dependence in the bound (64).

If $\phi_0 \leqslant 1$, then there is no dependence. Suppose therefore that $\phi_0 > 1$ and set $\bar{T} = \log \phi_0$. Then, for $l > \bar{T}$, $\phi_0 e^{-sl/2} < 1$. Thus the only contributions to estimate are those coming from $l < \bar{T}$. Using the fact that $q_l \leqslant n_l$, we will take account of half of the bound (66) and estimate

$$\prod_{l \leqslant \bar{T}} \phi_0^{q_l} e^{-(1/2) s l q_l} e^{-(q_l^2/2000)} \tag{68}$$

Let us fix $\sum_{l \leqslant \bar{T}} q_l = Q$ and compute the maximum of this expression. It is reached for

$$q_l = 500 \frac{\sum s_l}{\Delta} + \frac{Q}{\Delta} - 500 s_l$$

where $\Delta = \sum_{l \leqslant \bar{T}} 1$ is the sum over the l such that $q_l \neq 0$ and $1 \leqslant \Delta \leqslant \bar{T}$ if at least one $q_l \neq 0$. It is then easy to show that (68) is bounded by an expression of the form

$$e^{a + bQ - cQ^2} \leqslant e^{a + (b^2/4c)} = K(\bar{T})$$

a, b, and c are $\bar{T}$ dependent[3] and $c = 1/(2000\Delta) > 0$. Thus the contribution due to ϕ_0 can be bounded uniformly, for all terms of the expansion, by $K(\bar{T}) = K_5(\phi_0)$.

Therefore

$$\prod_{m=1}^{k} K(u_m, v_m)^2 I \leqslant K_3^2 K_5 (2KK_4)^{\Sigma n_l} \leqslant K_7 K_6^k$$

with $K_6 = (2KK_4)^{10}$ and $K_7 = K_5 K_3^2 (2KK_4)^{d^\circ F}$.

We now show how to get an exponential decrease in the length of the support of I. It is easy to show, using the fact that for $u_l \in \,]i_l, i_l + 1]$ and $v_l \in [j_l, j_l + 1]$,

$$|u_l - v_l| \geqslant i_l - j_l - 1$$

that

$$\prod_{l=1}^{k-1} e^{-|u_l - v_l|/8} \leqslant e^{-(1/8)(T - j_1 + \cdots + i_{k-1})} e^{(k-1)/8} \leqslant e^{-(T - j_{k-1})/8} e^{(k-1)/8}$$

We will use this result twice, first to control the Wick bound and second to make explicit an overall exponential decrease in the support.

Moreover, since either in an A or in a B term there is at least one functional derivative acting on the exponential, at least one new coupling constant λ can be explicitly factorized.

We will now show how, using one-quarter of each propagator, we can sum over all the possible time localizations.

The derivative with respect to s_l at the lth step gives rise to two sums over time localizations. One is related to the choice of a unit time interval $[i_l, i_l + 1]$, containing u_l, in the set $]j_{l-1}, T]$, and the other to the choice of a unit time interval $[j_l, j_l + 1]$, containing v_l, in $[0, j_{l-1}]$. Since by construction j_{l-1} is between u_l and v_l

$$|u_l - v_l| = |u_l - j_{l-1}| + |j_{l-1} - v_l| \leqslant |i_l - j_{l-1}| + |j_{l-1} - j_l|$$

[3] Since $\sum s_l$ and $\sum s_l^2$ are functions of $\bar{T}$.

Thus

$$K(v_l - u_l) \leqslant K(j_{l-1} - j_l)\, K(i_l - j_{l-1})$$

and we can use the fact that

$$\sum_{n \geqslant 0} K(n) \leqslant K_8$$

for some constant K_8, to sum over the localizations of u_l and of v_l, with j_{l-1} being fixed.

Therefore to perform the sums we start, using the above procedure, from the end, doing first the sums over the localizations of the derivatives with respect to s_{k-1}, then from the sums resulting from s_{k-2}, and so on, up to s_1.

We finally end up, for $k \geqslant 2$, with

$$\begin{aligned} I_k D_{k-1} \cdots D_1 E^T_{\phi_0} &\leqslant \lambda^{k-1} C_F K_7 K_6^k K_1^{k-1} K_2^{k-1} \sum_{i_1, j_1} \cdots \sum_{i_{k-1}, j_{k-1}} \prod_m K(u_m, v_m) \\ &\quad \times e^{-|T - j_{k-1}|/8} e^{(k-1)/8} e^{(6\lambda/8)(k+1) + (6\lambda/4)(T - j_{k-1})} \\ &\leqslant \lambda^{k-1} C_F K_7 K_8^{k-1} K_6^k K_1^{k-1} K_2^{k-1} \\ &\quad \times e^{(k-1)/8} e^{(6\lambda/8)(k+1)} \sup_{j_{k-1}} e^{-|T - j_{k-1}|/8} e^{(6\lambda/4)(T - j_{k-1})} \\ &\leqslant K\mu^{k-1} \end{aligned} \tag{69}$$

if $12\lambda < 1$, $\mu = \lambda K_1 K_2 K_6 K_8 e^{3\lambda/2}$ and $K = C_F K_6 K_7 e^{3\lambda/2}$.

Remark that only K_7 is ϕ_0 dependent. Therefore the constraint on the smallness of λ does not depend on ϕ_0.

We can obviously extend this bound to include the case $k = 1$, with K now such that $|I_1 E^T_{\phi_0}| < K$.

Thus we have proved that provided that $\mu < 1$, i.e., λ is small enough, $\lambda < \lambda_0$,

$$|E_{\phi_0}(F(\phi_T)\, e^{\xi_T})| < \frac{K}{1 - \mu} \tag{70}$$

with $K = K(\phi_0)$ finite for ϕ_0 finite.

5. PROOF OF THEOREM 1

The expansion defined in the previous section and the bound (70) prove the first assertion of Theorem 1. We will now show that from this we obtain the existence in a weak sense of an invariant measure and that this measure is insensitive to the initial conditions.

5.1. Existence of the Limit as a Convergent Expansion

To prove the second assertion of Theorem 1, we will show that $I_T = E_{\phi_0}(F(\phi_T)\, e^{\xi_T})$ satisfies the Cauchy criterion, i.e., that for any given $\varepsilon > 0$, there exists N such that for $T, T' \geqslant N$,

$$|I_T - I_{T'}| \leqslant \varepsilon$$

Thus let us suppose that $T > T'$. We then perform the expansions for both I_T and $I_{T'}$ and compare each term of I_T with the corresponding term, if it exists, of $I_{T'}$.

Let us look in detail at the first terms. Explicitly

$$E_{\phi_0}(F(\phi_T)\, e^{\xi_{T,T-1}}) = \int d\mu_C\, F(\phi_T) \times \exp\left(-\frac{\lambda}{4} \int_{T-1}^{T} :\phi_s^4:\, ds - \frac{\lambda^2}{8} \int_{T-1}^{T} (:\phi_s^3:)^2\, ds - \frac{\lambda}{8} :\phi_T^4: \right) \tag{71}$$

where the integration is over the Gaussian variable $\hat{\phi}_s$ (see eq. (63)) of mean 0 and covariance C. Let us now set $s = T - 1 + u$, then

$$\phi_s = e^{-(u/2)} \phi_{(T-1)} + \psi_u^{(T)} \tag{72}$$

with $\psi_u^{(T)} = \hat{\phi}_{T-1+u}$ and $\phi_{(T-1)} = e^{-(T-1)/2} \phi_0$. The interest of this change of variables comes from the fact that the covariance of ψ_u,

$$E(\psi_u^{(T)} \phi_v^{(T)}) = C(T-1+u, T-1+v) = e^{-|u-v|/2}(1 - e^{-(T-1)} e^{-\mathrm{Inf}(u,v)})$$

converges to the translation-invariant equilibrium Gaussian covariance when $T \to +\infty$.

Since in formula (71) all variables depend on times between T and $T-1$, u is restricted to the interval $[0, 1]$. In particular, the exponent can be rewritten as

$$\begin{aligned} \zeta(\phi_{(T-1)}) = & -\frac{\lambda}{4} \int_0^1 :(e^{-u/2} \phi_{(T-1)} + \psi_u^{(T)})^4:\, du \\ & - \frac{\lambda^2}{8} \int_0^1 (:(e^{-u/2} \phi_{(T-1)} + \psi_u^{(T)})^3:)^2\, du \\ & - \frac{\lambda}{8} :(e^{-1/2} \phi_{(T-1)} + \psi_1^{(T)})^4: \end{aligned} \tag{73}$$

Thus the T dependence of $E_{\phi_0}(F(\phi_T)\, e^{\xi_{T,T-1}})$ appears through the initial value of the variable $\phi_{(T-1)}$ and the propagator. We can therefore write the difference

$$E_{\phi_0}(F(\phi_T)\exp(\xi_{T,T-1})) - E_{\phi_0}(F(\phi_{T'})\exp(\xi_{T',T'-1})) = I_1(T, T')$$

using the fondamental formula of calculus, as the integral of the derivative of an interpolating formula obtained by replacing the initial condition by

$$\phi(\tau) = (e^{-(T-1)/2}\tau + e^{-(T'-1)/2}(1-\tau))\,\phi_0 \tag{74}$$

and the covariance C by

$$C_\tau(u, v) = e^{-|u-v|/2}(1 - (e^{-(T-1)}\tau + e^{-(T'-1)}(1-\tau))\, e^{-\mathrm{Inf}(u,v)})$$

Defining

$$I^{(1)}_{T,T'}(\tau) = \int d\mu_{C_\tau}\, F(e^{-1/2}\phi(\tau) + \psi_1)\, e^{\zeta(\phi(\tau))} \tag{75}$$

one has

$$I_1(T, T') = \int_0^1 \frac{d}{d\tau} I^{(1)}_{T,T'}(\tau)\, d\tau \tag{76}$$

The derivative $d/d\tau$ acts on $\phi(\tau)$, i.e., on functions of ϕ, F, or the exponent and on the measure. The result is a sum of two terms: the product of a derivative of $\phi(\tau)$ times the expectation value of some polynomial of ϕ and the derivative of the measure,

$$\int d\mu_{C_\tau} \iint (e^{-(T'-1)} - e^{-(T-1)})\, e^{-|u-v|/2}$$
$$\times \frac{\delta}{\delta\psi_u}\frac{\delta}{\delta\psi_v} F(e^{-1/2}\phi(\tau) + \psi_1)\, e^{\zeta(\phi(\tau))}\, du\, dv$$

Setting as before $\log\phi_0 = \bar{T}$ and supposing that T and T' are large enough such that

$$\inf(T, T') > 16\bar{T} \tag{77}$$

let us look at the first contribution. Since

$$\frac{d}{d\tau}\phi(\tau) = e^{-1/2}\phi_0(e^{-T/2} - e^{-T'/2})$$

and

$$|\phi_0(e^{-T/2} - e^{-T'/2})| \leqslant e^{-(7/16)\inf(T,T')}$$

one has

$$\left|\frac{d\phi(\tau)}{d\tau}\right| \leqslant e^{-1/2} e^{-(7/16)\inf(T,T')}$$

from which it follows that the first contribution is bounded by

$$\left|\frac{d}{d\tau} I^{(1)}_{T,T'}(\tau)\right| \leqslant e^{-(7/4)\inf(T,T')} \int e^{\zeta(\phi(\tau))} P(\phi(\tau))\, d\mu_{C_\tau}$$

where the polynomial $P(\phi(\tau))$ is given by

$$\begin{aligned} P(\phi(\tau)) &= e^{-1/2} F'(e^{-1/2}\phi(\tau) + \psi_1) \\ &\quad + F(e^{-1/2}\phi(\tau) + \psi_1)) \left\{ -\lambda \int_0^1 e^{-u/2} :(e^{-u/2}\phi(\tau) + \psi_u)^3: du \right. \\ &\quad - \frac{3\lambda^2}{4} \int_0^1 e^{-u/2} :(e^{-u/2}\phi(\tau) + \psi_u)^3: :(e^{-u/2}\phi(\tau) + \psi_u)^2: du \\ &\quad \left. - \frac{\lambda}{2} e^{-1/2} :(e^{-1/2}\phi(\tau) + \psi_1)^3: \right\} \end{aligned}$$

We can estimate this last term using the result of the previous section. The dependence with respect to the initial condition of this estimate can be removed by using the fact that, from condition (77),

$$|\phi(\tau)| \leqslant 1$$

Similarly, the second contribution is bounded by $|e^{-(T'-1)} - e^{-(T-1)}|$ times some uniformly bounded in T and T' expectation (according to the first assertion of Theorem 1), i.e., a constant times

$$e^{-(7/4)\operatorname{Inf}(T,T')}$$

This shows that

$$\lim_{\operatorname{Inf}(T,T') \to +\infty} I_1(T, T') = 0$$

and that

$$\lim_{T \to +\infty} E_{\phi_0}(F(\phi_T)\, e^{\xi_{T,T-1}}) = \int d\mu_{C_{st}} F(\phi_1)\, e^{\xi_1}$$

where ϕ_u is a Gaussian variable of mean 0 and covariance the translation-invariant equilibrium Gaussian covariance.

Let us suppose now that $T \geqslant T'$. To deal with a generic term, we follow the analysis done for the first term. Let us suppose first that at the kth step of the expansion one has a term with support $T - j_{k-1}$ for I_T with $j_{k-1} \geqslant T - T'$ and $T' - j'_{k-1}$ for I'_T such that $T' - j'_{k-1} = T - j_{k-1}$. Then, it is easy to convince oneself that these terms can be written as sums of elementary localized products of monomials corresponding to the same action of the derivatives and localized in unprimed and primed indices whose difference for the same labeling is $T - T'$. To compare these same elementary terms we perform a change of variable by setting $s = j_{k-1} + u$ and $s' = j'_{k-1} + u$ in, respectively, the unprimed and the primed expression. The difference between these two expressions is then related to the difference between the initial conditions. An interpolating initial condition is given by

$$\phi(\tau) = (e^{-j_{k-1}} - e^{-j'_{k-1}})\,\phi_0 \tau + e^{-j'_{k-1}}\phi_0$$

We have formulas identical to (75) and (76). The derivative with respect to τ produces two effects: first, an overall factor corresponding to the derivative of $\phi(\tau)$ which is bounded by

$$\left|\frac{d\phi(\tau)}{d\tau}\right| \leqslant |\phi_0(e^{-j_{k-1}} - e^{-j'_{k-1}})| \tag{78}$$

and second, contributions to the functional derivative from the chain rule

$$\frac{d}{d\tau} = \frac{d\phi(\tau)}{d\tau}\frac{\delta}{\delta\psi_u}$$

Obviously, one has for these terms, before we perform the sums over the localization indices $(i_1, j_1, \ldots)$, the same bounds as given in Section 4.3. Combining the overall decrease in the support, which was not used in the proof of the uniform estimate, with (78), one gets

$$\begin{aligned} e^{-(T - j_{k-1})/8}\left|\frac{d\phi(\tau)}{d\tau}\right| &\leqslant K_9\,|\phi_0|\,e^{-T'/8} = K_9(|\phi_0|\,e^{-\inf(T,T')/16})\,e^{-\inf(T,T')/16} \\ &\leqslant K_9 e^{-\inf(T,T')/16} \end{aligned}$$

In the case $j_{k-1} < T - T'$, the difference reduces to the terms in the expansion of I_T. But these terms, by the nature of the expansion, get a decrease in $e^{-(T - j_{k-1})/8} < e^{-T'/8}$.

Thus, collecting all results, we obtain for the difference an overall uniform bound in

$$e^{-\inf(T, T')/16}$$

Finally, this shows that there exists a constant K_{10} independent of λ, $\lambda < \lambda_0$, and ϕ_0 such that

$$|I_T - I_{T'}| \leqslant K_{10} e^{-\operatorname{Inf}(T, T')/16}$$

provided (77) is satisfied. This is the announced bound with $\varepsilon = K_{10} e^{-\operatorname{Inf}(T, T')/16}$.

5.2. Insensitivity to the Initial Condition

The third part of the theorem is proven in the same way by showing that

$$\lim_{T \to +\infty} \frac{d}{d\phi_0} (F(\phi_T)\, e^{\xi_T}) = 0$$

The insensitivity to the initial conditions then follows if we can interchange the limit with the derivative. It is trivial to see that this is possible for ϕ_0 in a compact.

Performing the derivative, we obtain

$$\frac{d}{d\phi_0} E_{\phi_0}(F(\phi_T)\, e^{\xi_T}) = e^{-T/2} E_{\phi_0}(F'(\phi_T)\, e^{\xi_T})$$
$$+ E_{\phi_0}\left(F(\phi_T))\, \frac{d\xi_T}{d\phi_0}\, e^{\xi_T}\right) \tag{79}$$

We can apply the expansion for the expectation in the first term. This shows that

$$\lim_{T \to \infty} E_{\phi_0}(F'(\phi_T)\, e^{\xi_T})$$

is bounded uniformly in T. Thus, because of the exponential factor, the first term of the r.h.s. of (79) vanishes as $T \to +\infty$.

We now look at the second term. We will then perform the expansion of Section 4 on this expression, but introducing the dependence with respect to the interpolating parameters also in the derived ξ_T:

$$E_{\phi_0}\left(F(\phi_T)\frac{d\xi_T}{d\phi_0}e^{\xi_T}\right)(s_1)$$

$$= E_{\phi_0}\left(F(\phi_T)\left(-\lambda\int_0^T e^{-s/2}:\phi_s^3:ds\right.\right.$$

$$= -\frac{3\lambda^2}{4}\int_0^T e^{-s/2}:\phi_s^3::\phi_s^2:ds - \frac{\lambda}{2}e^{-T/2}:\phi_T^3:$$

$$\left.\left.-(1-s_1)\frac{\lambda}{2}e^{-(T-1)/2}:\phi_{T-1}^3:+\frac{\lambda}{2}:\phi_0^3:\right)e^{\xi_T(s_1)}\right)$$

Thus

$$I_1 E_{\phi_0}\left(F(\phi_T)\frac{d\xi_T}{d\phi_0}e^{\xi_T}\right)$$

$$= I_1 E_{\phi_0}\left(F(\phi_T)\frac{d\xi_{T,T-1}}{d\phi_0}e^{\xi_T}\right) + I_1 E_{\phi_0}\left(F(\phi_T)\frac{d\xi_{T-1}}{d\phi_0}e^{\xi_T}\right) \tag{80}$$

We claim that the first term of the r.h.s. of (80) is exponentially small and that the second one vanishes.

Because of the support properties, we have

$$I_1 E_{\phi_0}\left(F(\phi_T)\frac{d\xi_{T,T-1}}{d\phi_0}e^{\xi_T}\right) = I_1 E_{\phi_0}\left(F(\phi_T)\frac{d\xi_{T,T-1}}{d\phi_0}e^{\xi_{T,T-1}}\right)$$

and it is easy to see that, performing the change of variables (72), this expression is bounded by, because of the factor $e^{-s/2}$ or $e^{-T/2}$ in the derivative of ξ,

$$I_1 E_{\phi_0}\left(F(\phi_T)\frac{d\xi_{T,T-1}}{d\phi_0}e^{\xi_{T,T-1}}\right) \leqslant K'_{11}e^{-(T-1)/2} < K_{11}e^{-T/2}$$

with K_{11} independent of T.

The second term factorizes as

$$E_{\phi_0}\left(F(\phi_T)\frac{d\xi_{T-1}}{d\phi_0}e^{\xi_T}\right) = E_{\phi_0}(F(\phi_T)\,e^{\xi_{T,T-1}})\,E_{\phi_0}\left(\frac{d\xi_{T-1}}{d\phi_0}e^{\xi_{T-1}}\right)$$

but

$$E_{\phi_0}\left(\frac{d\xi_{T-1}}{d\phi_0}e^{\xi_{T-1}}\right) = \frac{d}{d\phi_0}E_{\phi_0}(e^{\xi_{T-1}}) = \frac{d}{d\phi_0}1 = 0$$

There remains a D_1 term to which we apply the next step of the expansion. We consider the A term. Thus

$$A_1 E_{\phi_0}\left(F(\phi_T)\frac{d\xi_T}{d\phi_0}e^{\xi_T}\right)$$

$$=\sum_{j_1=0}^{T-1} A_1(j_1+1)=\sum_{j_1=0}^{T-1}\int_0^1\int d\mu_{C(s_1)}\int C(u_1,v_1)\,\chi_{j_1+1}(u_1)\,\chi_T(v_1)$$

$$\times\frac{\delta}{\delta\phi_{u_1}}\frac{\delta}{\delta\phi_{v_1}}\left(F(\phi_T)\frac{d\xi_T(s_1)}{d\phi_0}e^{\xi_T(s_1)}\right)du_1\,dv_1\,ds_1 \tag{81}$$

We introduce the s_2 dependence as in Section 4 in $\xi_T(s_1)$ and in $d\xi_T(s_1)/d\phi_0$.

Then its contribution to

$$I_2 D_1 E_{\phi_0}\left(F(\phi_T)\frac{d\xi_T}{d\phi_0}e^{\xi_T}\right)$$

is given by

$$I_2 A_1(j_1+1)$$

$$=\int_0^1\int d\mu_{C(s_1)}\int C(u_1,v_1)\,\chi_{j_1}(u_1)\,\chi_T(v_1)$$

$$\times\frac{\delta}{\delta\phi_{u_1}}\frac{\delta}{\delta\phi_{v_1}}\left(F(\phi_T)\frac{d\xi_T(s_1,s_2)}{d\phi_0}e^{\xi_T(s_1,s_2)}\right)du_1\,dv_1\,ds_1\,|_{s_2=0}$$

$$=\int_0^1\int d\mu_{C(s_1)}\int C(u_1,v_1)\,\chi_{j_1}(u_1)\,\chi_T(v_1)$$

$$\times\frac{\delta}{\delta\phi_{u_1}}\frac{\delta}{\delta\phi_{v_1}}\left(F(\phi_T)\frac{d\xi_{T,j_1}(s_1,0)}{d\phi_0}e^{\xi_{T,j_1}(s_1,0)}\right)du_1\,dv_1\,ds_1$$

$$+\int_0^1\left\{\int d\mu_{C(s_1)}\int C(u_1,v_1)\,\chi_{j_1}(u_1)\,\chi_T(v_1)\right.$$

$$\left.\times\frac{\delta}{\delta\phi_{u_1}}\frac{\delta}{\delta\phi_{v_1}}(F(\phi_T)\,e^{\xi_{T,j_1}})\,du_1\,dv_1\right\}$$

$$\times\left\{\int d\mu_{C(s_1)}\int C(u_1,v_1)\,\chi_{j_1}(u_1)\,\chi_T(v_1)\right.$$

$$\left.\times\frac{\delta}{\delta\phi_{u_1}}\frac{\delta}{\delta\phi_{v_1}}\left(\frac{d\xi_{j_1}}{d\phi_0}e^{\xi_{j_1}}\right)du_1\,dv_1\right\}ds_1$$

The first term on the r.h.s. can be estimated by the method of Section 4. Taking account of the extra exponential factor due to the derivative with respect to ϕ_0, we get a bound of the form (for suitable K and μ)

$$K\mu e^{-j_1/2} e^{-(T-j_1)/8} \leqslant K\mu e^{-T/8} e^{-3j_1/8}$$

We thus get an extra factor $e^{-T/8}$. There is a similar bound for the B term.

The second term is identically 0 since the second factor in the product is a derivative of the identity.

These results can be extended for a general term of the kth order showing that

$$E_{\phi_0}\left(F(\phi_T)\frac{d\xi_T}{d\phi_0}e^{\xi_T}\right)$$

is bounded by a convergent sum times $e^{-T/8}$. Thus when $T \to \infty$ this contribution vanishes. This ends the proof of the theorem.

6. PROOF OF THEOREM 2

6.1. Existence of the Measure

The existence of the limit

$$\lim_{T \to \infty} E_{\phi_0}(F_1(\phi_{T+t_1}) \cdots F_n(\phi_{T+t_n})\, e^{\xi_{T+t}})$$

is proved in the same way one prove the existence of the limit for a single-time expectation. Remark that this limit, if it exists, is the same as

$$\lim_{T' \to \infty} E_{\phi_0}(F_1(\phi_{T'-\alpha+t_1}) \cdots F_n(\phi_{T'-\alpha+t_n})\, e^{\xi_{T'-\alpha+t}})$$

for any finite α. This show that the limit of the expectation of correlation functions is translation invariant.

The proof follows essentially the steps developed in Sections 4 and 5.1., except that the initial interval is no longer $[T-1, T]$, but $[T, T+t]$, i.e., we first write the initial expression as a uniformly in T convergent expansion and then show the existence of the limit by a three-ε argument.

The existence of the limiting measure can be proved by different methods. One possible method is to show that for any function in L^2 with support in $[0, t]$, for some $t > 0$, there exists the characteristic function

$$E_{\phi_0}\left(\exp\left\{i\int f(s-T)\,\phi_s\,ds\right\}\exp(\xi_T)\right)$$

6.2. Exponential Clustering

The exponential clustering follows also from the same analysis. We will prove this result for the expectation of two variables. Let us consider, with $t_1 \geqslant t_2 + 1$,

$$I(t_1, t_2) = \lim_{T \to \infty} E_{\phi_0}(F_1(\phi_{T+t_1}) F_2(\phi_{T+t_2}) e^{\xi T + t_1}) - [\lim_{T \to \infty} E_{\phi_0}(F_1(\phi_{T+t_1}) e^{\xi T + t_1})][\lim_{T \to \infty} E_{\phi_0}(F_2(\phi_{T+t_2}) e^{\xi T + t_1})] \tag{82}$$

We rewrite this expression as

$$I(t_1, t_2) = \lim_{T \to \infty} \{E_{\phi_0}(F_1(\phi_{T+t_1}) F_2(\phi_{T+t_2}) e^{\xi T + t_1}) - E_{\phi_0}(F_1(\phi_{T+t_1}) e^{\xi T + t_1})[\lim_{T' \to \infty} E_{\phi_0}(F_2(\phi_{T'+t_2}) e^{\xi T' + t_1})]\} \tag{83}$$

and apply the expansion of Section 4 to all the T-dependent terms of the r.h.s. of formula (83). The term $I_1 I(t_1, t_2)$ is then given by

$$I_1 I(t_1, t_2) = \lim_{T \to \infty} \{E_{\phi_0}(F_1(\phi_{T+t_1}) e^{\xi T + t_1, T + t_1 - 1}) E_{\phi_0}(F_2(\phi_{T+t_2}) e^{\xi T + t_1 - 1})\} - [\lim_{T \to \infty} E_{\phi_0}(F_1(\phi_{T+t_1}) e^{\xi T + t_1, T + t_1 - 1})] \times [\lim_{T' \to \infty} E_{\phi_0}(F_2(\phi_{T'+t_2}) e^{\xi T' + t_2})] \tag{84}$$

Because of the Markov property, we have

$$E_{\phi_0}(F_2(\phi_{T+t_2}) e^{\xi T + t_1 - 1}) = E_{\phi_0}(F_2(\phi_{T+t_2}) e^{\xi T + t_2})$$

and because individually each term has a limit, one gets that

$$I_1 I(t_1, t_2) = 0$$

Let us consider now the second term of the expansion,

$$I_2 D_1 I(t_1, t_2) = \lim_{T \to \infty} \{I_2 D_1 E_{\phi_0}(F_1(\phi_{T+t_1}) F_2(\phi_{T+t_2}) e^{\xi T + t_1}) - I_2 D_1 E_{\phi_0}(F_1(\phi_{T+t_1}) e^{\xi T + t_1}) \times [\lim_{T' \to \infty} E_{\phi_0}(F_2(\phi_{T'+t_2}) e^{\xi T' + t_1})]\} \tag{85}$$

Each $I_2 D_1$ gives a sum of contribution labeled by some index j_1.

If $j_1 \geqslant T+t_2$, then

$$I_2 D_1 E_{\phi_0}(F_1(\phi_{T+t_1})\, F_2(\phi_{T+t_2})\, e^{\xi_{T+t_1}})$$
$$= D_1 E_{\phi_0}(F_1(\phi_{T+t_1})\, e^{\xi_{T+t_1, j_1}})\, E_{\phi_0}(F_2(\phi_{T+t_2})\, e^{\xi_{j_1}})$$

Thus again, because of the Markov property, the contributions indexed by j_1 to (85) vanish when $T \to \infty$.

When $j_1 < T+t_2$, then each contribution from $I_2 D_1$ gives an extra factor

$$e^{-(T-j_1)/8} \leqslant e^{(t_1-t_2)/8}$$

Repeating this type of argument shows that

$$I(t_1, t_2) \leqslant K e^{-(t_1-t_2)/8}$$

We have thus proved our assertion.

7. NONGRADIENT SYSTEMS AND MORE GENERAL POTENTIAL FUNCTIONS

All the calculations and results of the previous sections can be trivially extended with obvious changes to the case when the stochastic variables **X** are finite-dimensional vectors and the matrix A is nonsymmetric but has spectrum σ whose real part is strictly positive. In this case, the covariance C given by (17) satisfies the inequality

$$\|C(t_1, t_2)\| \leqslant \text{Const}\ \|M\| \exp(-\min \Re\sigma_i\, |t_1 - t_2|) \tag{86}$$

where $\|\cdot\|$ is an operator norm in the space of the vector variable **X** and $\Re\sigma_i$ is the real part of the eigenvalue σ_i. All the estimates go through with minor changes.

The expansion and the results have been given for pedagogical reasons for one-dimensional systems with a polynomial drift. The reason for the choice of polynomial drifts comes from the fact that when working with SPDE, i.e., the random variables are now $\phi_s(x)$, indexed by a space parameter $x \in R^n$, $n = 1, 2,\ldots$, unbounded functions like polynomials are a good prototype of the main difficulties one will encounter. In analogy with quantum field theories, arbitrary polynomials are allowed for $n=0$ and 1, Wick-ordered polynomials for $n=2$, and very special polynomials for $n \geqslant 3$, since one has to face nontrivial renormalization problems in this case.

For SODE, from the nature of the expansion, it is easy to see that the proof extends to local potentials $V(\phi_s)$ where V is a C_∞ function whose derivatives satisfy bound like

$$|V^{(r)}(\phi)| \leqslant K(r!)^\alpha \, \phi^{2\beta_r} \qquad \forall r \in Z^+$$

where K is a constant and α and β are some positive integers independent of r. Such a bound will not destroy the convergence, because the expectation of the monomials will produce an $(r!)^\beta$ giving rise to a local $(r!)^{\alpha+\beta}$, which in turn can be controlled like the usual local factorials (see Section 4.3).

All the previous arguments hold in case the initial condition ϕ_0 is averaged with respect to a Gaussian-like probability measure, i.e., a measure $\nu(\phi_0)$ such that for all n

$$\int |\phi_0|^n \, d\nu(\phi_0) < C n^{\alpha n}$$

where C and α are positive constants.

8. REMARKS ON THE STRUCTURE OF THE EXPANSION FOR THE STATIONARY MEASURE

We wish to emphasize that the construction of the invariant measure via the cluster expansion is a subtle process. This can be well illustrated in the case of the one-dimensional model considered in the previous section. Take the first term of the expansion given by Eq. (32),

$$E^{S_1}(F(\phi_1)\, e^{\xi_1})$$

and suppose we want to evaluate it by making an expansion in λ. To first order we have

$$E^{S_1}(F(\phi_1)) - \frac{\lambda}{8} E^{S_1}(F(\phi_1) :\phi_1^4:) - \frac{\lambda}{4} E^{S_1}\left(F(\phi_1) \int_0^1 :\phi_s^4:\, ds\right) + O(\lambda^2) \quad (87)$$

Let us take $F(\phi) = :\phi^4:$ to have a nonzero contribution at this order. Then to order λ we obtain

$$E^{S_1}(F(\phi_1)\, e^{\xi_1}) = -\frac{\lambda}{8} 4![1 + 1(1 - e^{-2})] + O(\lambda^2) \quad (88)$$

On the other hand, calculating to the same order with the invariant measure μ_{Inv}, one has

$$\int d\mu_{\text{Inv}}(\phi)\, F(\phi) = -\frac{\lambda}{4}4! + O(\lambda^2) \tag{89}$$

One sees that the first two terms of the Girsanov exponent cooperate to give almost the right coefficient. This means that the defect must be compensated by other terms of the cluster expansion. Terms of order λ appear only in A_1 and B_1 so that the compensating term will come from $I_2(A_1 + B_1)$. This can be verified by direct calculation.

In the nongradient case, when the matrix A in Eq. (3) is nonsymmetric, the two terms of the Girsanov exponent appearing in (87) will have a very different structure. The first one depends only on M, while the second one depends explicitly on A.

It is instructive to give their expression for a simple system. Let us take V of the form

$$V(\boldsymbol{\phi}) = \tfrac{1}{8} \sum_{i=1}^{n} :\phi_i^4: \tag{90}$$

where

$$:\phi_i^4: = \phi_i^4 - 6M_{ii}\phi_i^2 + 3M_{ii}^2 \tag{91}$$

To order λ, we have from the first term of the cluster expansion, taking $F(\phi) = :\phi_i^4:$ to have a nonzero contribution at this order,

$$\begin{aligned} &-\frac{\lambda}{8} E^{\text{St}}\Big(:\phi_{i1}^4: \sum_k :\phi_{k1}^4: \Big) - \frac{\lambda}{2} \sum_{j,k} E^{\text{St}}\Big(:\phi_{i1}^4: \int_0^1 ds\, :\phi_{js} A_{jk}^T \phi_{ks}^3: \Big) \\ &\quad = -\frac{\lambda}{8} 4! \sum_k M_{ik}^4 - \frac{\lambda}{2} 4! \sum_{j,k} \int_0^1 ds\, C_{ij}(1,s)\, C_{ik}^3(1,s)\, A_{jk}^T \end{aligned} \tag{92}$$

where

$$C_{ik}(1,s) = (e^{-A(1-s)}M)_{ik} \tag{93}$$

For a generic A there is no simple connection between the two terms appearing in (92).

We now consider a different aspect which is relevant in concrete calculations. If we examine the first terms of the expansion $I_1 E + I_2 A_1 + I_2 B_1$, we notice that $I_1 E$ and $I_2 B_1$ involve averages over trajectories defined on finite time intervals, while $I_2 A_1$ has been decomposed into an infinite sum of terms involving unit intervals separated by increasing times. The contributions from distant intervals, however, decrease exponentially due to the decay of the Ornstein–Uhlenbeck covariance, which provides the relevant relaxation time scale. This is interesting because if we compute these contributions with the help of numerical simulation, we have a precise criterion to decide its duration on the basis of the accuracy we desire. This argument of course applies also to higher order terms in the cluster expansion.

We conclude by citing two very recent papers where cluster expansion ideas are used in the study of time evolutions, of which we became aware after the completion of the present work.[11,12]

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REFERENCES

1. G. Jona-Lasinio and P. K. Mitter, On the stochastic quantization of field theory, *Commun. Math. Phys.* **101**:401 (1985).
2. G. Jona-Lasinio and P. K. Mitter, Large deviation estimates in the stochastic quantization of ϕ_2^4, *Commun. Math. Phys.* **130**:111 (1990).
3. G. Jona-Lasinio and R. Seneor, On a class of stochastic reaction-diffusion equations in two space dimensions, *J. Phys. A: Math. Gen.* **24**:4123 (1991).
4. R. S. Liptser and A. N. Shiryaev, *Statistics of Stochastic Processes I* (Springer-Verlag, Berlin, 1977), Chapter 7.
5. D. Zwanziger, *Stochastic Quantization of Gauge Fields*, G. Velo and A. S. Wightman, eds. (Plenum Press, New York, 1986).
6. R. Z. Hasminski, *Stochastic Stability of Differential Equations* (Sijthoff & Noordhoff, 1980).
7. J. Glimm and A. Jaffe, *Quantum Physics*, 2nd ed. (Springer-Verlag, Berlin, 1987).
8. J. Magnen and R. Seneor, The infinite volume limit of the Φ_3^4 model, *Ann. Inst. H. Poincaré* **24**:95 (1976).
9. D. Brydges, A short course in cluster expansions in *Critical Phenomena, Random Systems, Gauge Theories (Les Houches 1984)*, K. Osterwalder and R. Stora, eds. (North-Holland, Amsterdam, 1986).
10. V. A. Malyshev and R. A. Minlos, *Gibbs Random Fields* (Kluwer, Dordrecht, 1991).
11. V. V. Scherbakov, Full-space exponential convergence for Ito stochastic partial differential equations, *Selecta Math.* **12**:223 (1993).
12. J. Bricmont and A. Kupiainen, Coupled analytic maps, Preprint (1994).